

RISK-SENSITIVE CONTROL OF CONTINUOUS TIME MARKOV CHAINS

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ABSTRACT. We study risk-sensitive control of continuous time Markov chains taking values in discrete state space. We study both finite and infinite horizon problems. In the finite horizon problem we characterise the value function via HJB equation and obtain an optimal Markov control. We do the same for infinite horizon discounted cost case. In the infinite horizon average cost case we establish the existence of an optimal stationary control under certain Lyapunov condition. We also develop a policy iteration algorithm for finding an optimal control.

1. Introduction and Preliminaries

In the last two decades considerable attention has been given to the investigation of risk sensitive problems in the literature of stochastic dynamic optimization. An important reason for the popularity of this kind of problems is its connections with H_∞ or robust control problems and stochastic dynamic games. A justification for the term risk-sensitive control comes from utility theory in economics. Generally in stochastic dynamic optimization, the decision maker (controller) seeks to minimise a cost functional which is a random quantity, say, X , which depends on the time horizon and the control adopted by the controller. Since X is random the controller tries to minimise the expected value of X . This is the risk neutral case. But this approach has some limitations namely if the variance is large then there can be issues with the optimal control. Generally variance is a measure of risk in economics literature. So ideally one would like to minimise both mean and variance simultaneously, but this may not be feasible. Therefore a convex combination of mean and variance is optimised or the mean is optimised for a given variance. This approach of mean-variance optimization was taken by Markowitz in his work on portfolio selection [20]. This was later extended by Sharpe in his capital asset pricing model [26]. But if the random variable is not normally distributed, then its distribution is not completely determined by the first two moments. Thus it is reasonable to consider a cost criterion which deals with higher moments as well. A powerful approach in this direction is the risk-sensitive control wherein the controller seeks to minimise an exponential criterion. Roughly speaking the cost functional of interest is of the form $\mathbb{E} \exp(\theta X)$ where X is the random variable which

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denotes the cost payable by the controller and $\theta > 0$ is a parameter chosen by the controller and whose interpretation is given below. Let w be the amount the controller is willing to pay instead of the random quantity X . Thus w satisfies

$$\exp(\theta w) = \mathbb{E} \exp(\theta X).$$

The deterministic quantity w is referred to as the certainty equivalent of X . The risk premium π is defined by the equation

$$w = \mathbb{E}X + \pi.$$

Now by Jensen's inequality

$$\exp(\theta \mathbb{E}X) \leq \mathbb{E} \exp(\theta X) = \exp(\theta w).$$

Thus by the monotonicity property of the exponential function, $w \geq \mathbb{E}X$, which implies $\pi \geq 0$. Thus in this case the controller is risk averse. Now to measure the degree of risk aversion, let $x = \mathbb{E}X$. Formally by Taylor's expansion

$$\exp(\theta w) = \exp(\theta x) + \pi \theta \exp(\theta x) + o(\pi).$$

Again

$$\mathbb{E} \exp(\theta X) = \exp(\theta x) + \frac{1}{2} \text{var}(X) \theta^2 \exp(\theta x) + \mathbb{E}(o(X - x)^2).$$

Thus we have $\pi = \frac{1}{2} \text{var}(X) \theta$ plus smaller order terms. Hence the risk premium is proportional to θ up to first order. That is why θ is referred to as the absolute risk aversion parameter. Similar arguments can also be made for $\theta < 0$ case. In that case the controller is risk seeking. The limiting case of $\theta = 0$ is the risk neutral case.

There is a vast literature on the risk neutral case, for example see [1] for controlled diffusions, [11] for continuous time MDP, [13] for discrete time MDP and the references therein. See also [25] for variance minimization and overtaking optimality of continuous-time MDP. For earlier works on risk-sensitive control we refer to [15] and [16]. Since then there has been a lot of research on risk sensitive control of discrete time Markov chains [6], [7], [14], [17] [21] and also there has been a lot of work on risk sensitive control of diffusions [3], [9], [22], [23], [27]. As is evident from the discussion above, risk sensitive control has wide applications in economics and in particular in finance [10], [4], [5], [24].

Although risk sensitive control of continuous time diffusions and discrete time Markov chains has been studied, the problem for continuous time MDP does not seem to have been studied in literature. In this paper we study risk-sensitive control of continuous time Markov chains. We take the state space S to be countable. For notational simplicity we take $S = \{0, 1, 2, \dots\}$. Let $U_i, i = 0, 1, \dots$ be compact metric spaces; U_i is the control set when the state is i . We denote the state process by $\{X_t\}$ and the control process by $\{U_t\}$.

Formally the dynamics of the process is as follows:

$$\begin{cases} \mathbb{P}(X_{t+h} = j \mid X_t = i, U_t = u) = \lambda_{ij}(u)h + o(h) \\ \mathbb{P}(X_{t+h} = i \mid X_t = i, U_t = u) = 1 - (\sum_{j \neq i} \lambda_{ij}(u))h + o(h), \end{cases} \quad (1.1)$$

where $\lambda_{ij} : U_i \rightarrow \mathbb{R}_+$ are given functions. That is, if the process is at i at time t and if the action chosen at that moment is u , then after a little while h the process will be at state j with probability $\lambda_{ij}(u)h$ plus some error term and the process will remain at i with probability $1 - (\sum_{j \neq i} \lambda_{ij}(u))h$ plus some error term. Thus λ_{ij} s are the instantaneous transition rates. Set

$$\lambda_{ii}(u) = - \sum_{j \neq i} \lambda_{ij}(u). \quad (1.2)$$

The following assumptions will be in force throughout the paper:

(A1) The function λ_{ij} s are continuous.

(A2) $\sup_i \sup_{u \in U_i} \{-\lambda_{ii}(u)\} \leq M < \infty$.

(A3) The sum in (1.2) converges uniformly. Thus λ_{ii} is continuous for each i .

We now describe a rigorous construction of the process $\{X_t\}$ via the martingale problem. Let $\mathcal{D} = \mathcal{D}([0, \infty), S)$ be the space of S -valued right-continuous functions with left limits endowed with the Skorokhod topology. Let \mathcal{S} be the Borel σ -algebra on \mathcal{D} . Define $U = \cup_i U_i$. Let $\mathbf{u} : [0, \infty) \times S \rightarrow U$ be such that $\mathbf{u}(\cdot, i) \in U_i$ and is measurable for each i . Let $B(S)$ denote the space of bounded real valued functions on S . For $f \in B(S)$, $\|f\|$ denotes the supremum norm. For each $t \in [0, \infty)$ define the operator $\Lambda_t^{\mathbf{u}} : B(S) \rightarrow B(S)$ by

$$\Lambda_t^{\mathbf{u}} f(i) = \sum_j \lambda_{ij}(\mathbf{u}(t, i)) f(j). \quad (1.3)$$

On the measurable space $(\mathcal{D}, \mathcal{S})$, let $\{X_t, t \geq 0\}$ denote the canonical process, i.e., for $\omega \in \mathcal{D}$, $X_t(\omega) = \omega(t)$. Let μ be any probability measure on S . The martingale problem corresponding to $(\Lambda^{\mathbf{u}}, \mu)$ is the following: A measure $\mathbb{P}_{s, \mu}^{\mathbf{u}}$ on $(\mathcal{D}, \mathcal{S})$ is said to be a solution for the martingale problem corresponding to $(\Lambda^{\mathbf{u}}, \mu)$ if

- i) $\mathbb{P}_{s, \mu}^{\mathbf{u}}(X_s \in A) = \mu(A)$ for any Borel subset A of S ;
- ii) $f(X_t) - \int_0^t \Lambda_s^{\mathbf{u}} f(X_s) ds$ is a $\mathbb{P}_{s, \mu}^{\mathbf{u}}$ martingale with respect to the filtration $\mathcal{F}_t = \sigma(X_r; r \leq t)$ for each f in $B(S)$.

Under **(A2)** it can be shown following the arguments in Chapter 6 of [8] that the above martingale problem has a unique solution and $\{X_t\}$ is a Markov process with the generator given by (1.3). In fact we can relax the boundedness condition in **(A2)**. If λ_{ii} s satisfy appropriate growth condition then also the martingale problem is well posed; see Chapter 6 of [8]. Also see [12] and the references therein for related works. From now on we will work in the canonical space $(\mathcal{D}, \mathcal{S})$. If $s = 0$ and $\mu = \delta_i$ for some $i \in S$ then we will write $\mathbb{P}_{s, \mu}^{\mathbf{u}}$ as $\mathbb{P}_i^{\mathbf{u}}$. The corresponding expectation operator is denoted by $\mathbb{E}_i^{\mathbf{u}}$. In our paper the set of admissible controls is the set of Markov controls, i.e., controls of the form $U_t = \mathbf{u}(t, X_{t-})$, for some $\mathbf{u} : [0, \infty) \times S \rightarrow U$, such that $\mathbf{u}(\cdot, i) \in U_i$ and is measurable for each i . With an

abuse of terminology the map \mathbf{u} itself is referred to as a Markov control. Let \mathcal{U} denote the set of all Markov controls. A Markov control is said to be stationary if the function \mathbf{u} has no explicit dependence on t , i.e., $\mathbf{u} : S \rightarrow U$, such that $\mathbf{u}(i) \in U_i$ for each i . The set of stationary Markov controls is denoted by \mathcal{U}_s .

Now we briefly describe the problems we consider in this paper. In stochastic dynamic optimization based on the time horizon there can be two kinds of problems namely finite horizon and infinite horizon problems. In this paper we address both infinite and finite horizon problems.

Finite Horizon Problem: Define $K = \{(i, u) : i \in S, u \in U_i\}$. Let $c : [0, \infty) \times K \rightarrow [0, \infty)$ be a bounded function, such that $c(\cdot, i, \cdot)$ is continuous for each i and $g : S \rightarrow [0, \infty)$ a bounded function. Let $0 < T < \infty$ be the length of the time horizon. Then for any Markov control U consider the cost functional

$$J_T^{\mathbf{u}}(i) = \frac{1}{\theta} \log \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \left[\int_0^T c(s, X_s, U_s) ds + g(X_T) \right] \right) \right] \quad (1.4)$$

for some $\theta \in (0, 1)$ and where $U_t = \mathbf{u}(t, X_{t-})$. In literature c is referred to as the running cost function and g as the terminal cost function. The aim of the controller is to minimise $J_T^{\mathbf{u}}$ over all Markov controls \mathbf{u} . A control $\hat{\mathbf{u}}$ is said to be optimal if

$$J_T^{\hat{\mathbf{u}}}(i) = \inf_{\mathcal{U}} J_T^{\mathbf{u}}(i).$$

Infinite Horizon Discounted Cost Problem: For the infinite horizon problems the running cost function has no explicit time dependence. For each Markov control \mathbf{u} define

$$I_\alpha(\theta, i, \mathbf{u}) = \frac{1}{\theta} \log \left(E_i^{\mathbf{u}} \left(\exp \left[\theta \int_0^\infty e^{-\alpha t} c(X_t, U_t) dt \right] \right) \right) \quad (1.5)$$

where θ is as in the finite horizon problem and $\alpha > 0$ is the discount factor. Here the controller wants to minimise $I_\alpha(\theta, i, \mathbf{u})$ over all Markov controls \mathbf{u} . A control $\hat{\mathbf{u}}$ is said to be optimal if it satisfies

$$I_\alpha(\theta, i, \hat{\mathbf{u}}) = \inf_{\mathcal{U}} I_\alpha(\theta, i, \mathbf{u}).$$

Infinite Horizon Average Cost Problem: For the average cost problem the set of admissible controls is the set of stationary Markov controls. For a stationary control \mathbf{u} , define

$$J^{\mathbf{u}}(i) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^T c(X_t, \mathbf{u}(X_t)) dt \right) \right]. \quad (1.6)$$

The controller wants to minimise $J^{\mathbf{u}}(i)$ over all stationary controls \mathbf{u} . Optimal control is defined analogously.

The rest of the paper is organised as follows. In Section 2 we study the finite horizon problem. This analysis of this problem is fairly straightforward. Using the dynamic programming heuristics we derive the Hamilton Jacobi Bellman (HJB) equation for this criteria. Then using a fixed point theorem and some standard arguments involving Dynkin's formula we

show that the value function is the unique solution of the HJB equation in an appropriate class of functions. This in turn yields the existence of an optimal Markov control. Section 3 deals with infinite horizon discounted cost case. The analysis of this problem is surprisingly far more involved from a technical view point. As usual by using the dynamic programming heuristics we derive the HJB equation and establish the corresponding verification theorem. However, to establish the existence of a smooth solution of the HJB equation for this criteria turns out to be quite tricky. We work around this problem by an appropriate limiting procedure which establishes a solution of the HJB equation in the sense of distributions. Then under certain assumptions we establish the desired regularity of the solution. In Section 4 we investigate the average cost problem. Again this problem turns out to be technically involved. The traditional vanishing discount approach does not seem to work. Instead we use the multiplicative Poisson equation to get at the desired result. Using a limiting argument involving the multiplicative Poisson equation we establish the existence of an optimal stationary control. In Section 5 we give a policy improvement algorithm for the average cost case. Finally in Section 6 we conclude our paper with some concluding remarks.

2. Finite Horizon Case

In this section we study the finite horizon case. For this we first study the exponential cost criterion. For $t \in [0, T]$, $\mathbf{u} \in \mathcal{U}$, define

$$\hat{J}_T^{\mathbf{u}}(t, i) = \mathbb{E}_{t,i}^{\mathbf{u}} \left[\exp \left(\theta \left[\int_t^T c(s, X_s, U_s) ds + g(X_T) \right] \right) \right]. \quad (2.1)$$

Define the value function V_T by

$$V_T(t, i) = \inf_{\mathcal{U}} \hat{J}_T^{\mathbf{u}}(t, i)$$

where the infimum is over all Markov controls. Our aim is to characterise the value function and to obtain an optimal control. To this end we first describe a heuristic derivation of the Hamilton Jacobi Bellman (HJB) equation. Formally

$$\begin{aligned} V_T(t, i) &= \inf_{\mathcal{U}} \mathbb{E}_{t,i}^{\mathbf{u}} \left\{ \exp \left[\theta \int_t^{t+h} c(s, X_s, U_s) ds + \theta \int_{t+h}^T c(s, X_s, U_s) ds + \theta g(X_T) \right] \right\} \\ &= \inf_{\mathcal{U}} \mathbb{E}_{t,i}^{\mathbf{u}} \left\{ \exp \left[\theta \int_t^{t+h} c(s, X_s, U_s) ds \right] \mathbb{E}_{t+h, X_{t+h}}^{\mathbf{u}} \left(\exp \left[\theta \int_{t+h}^T c(s, X_s, U_s) ds + \theta g(X_T) \right] \right) \right\} \\ &= \inf_{\mathcal{U}} \mathbb{E}_{t,i}^{\mathbf{u}} \left\{ \exp \left[\theta \int_t^{t+h} c(s, X_s, U_s) ds \right] V_T(t+h, X_{t+h}) \right\}. \end{aligned}$$

If the function $V_T(\cdot, i)$ is continuously differentiable then standard dynamic programming arguments involving Dynkin's formula leads to the following HJB equation for the finite horizon problem:

$$\begin{cases} \frac{d\varphi}{dt} + \inf_{u \in U_i} [\theta c(t, i, u) \varphi(t, i) + \sum_j \lambda_{ij}(u) \varphi(t, j)] = 0 \text{ on } [0, T) \times S \\ \varphi(T, i) = e^{\theta g(i)}. \end{cases} \quad (2.2)$$

The importance of this equation is highlighted by the following verification theorem:

Theorem 2.1. *Assume (A1)-(A3). Suppose there exists a smooth (continuously differentiable with respect to the first variable), bounded solution Ψ to (2.2), then*

$$\Psi(t, i) = V_T(t, i) \text{ for all } (t, i) \in [0, T] \times S.$$

Furthermore an optimal Markov control for the cost criterion (2.1) exists and is given by $U_t^* = \mathbf{u}^*(t, X_{t-})$ where \mathbf{u}^* satisfies

$$\begin{aligned} & \inf_{u \in U_i} [\theta c(t, i, u) \Psi(t, i) + \sum_j \lambda_{ij}(u) \Psi(t, j)] \\ &= [\theta c(t, i, \mathbf{u}^*(t, i)) \Psi(t, i) + \sum_j \lambda_{ij}(\mathbf{u}^*(t, i)) \Psi(t, j)]. \end{aligned} \quad (2.3)$$

Proof. Let \mathbf{u} be any arbitrary Markov control. By Feynman - Kac formula

$$\begin{aligned} & \mathbb{E}_{t,i}^{\mathbf{u}} \left[\exp \left(\theta \left[\int_t^T c(s, X_s, U_s) ds + g(X_T) \right] \right) \right] \\ &= \Psi(t, i) + \mathbb{E}_{t,i}^{\mathbf{u}} \int_t^T \exp \left(\theta \left[\int_t^r c(s, X_s, U_s) ds \right] \right) \left[\frac{d\Psi}{dr}(r, X_r) + \theta c(r, X_r, U_r) \Psi(r, X_r) + \sum_j \lambda_{X_r j}(U_r) \Psi(r, j) \right] dr. \end{aligned}$$

Since Ψ satisfies (2.2) we have

$$\Psi(t, i) \leq \mathbb{E}_{t,i}^{\mathbf{u}} \left[\exp \left(\theta \left[\int_t^T c(s, X_s, U_s) ds + g(X_T) \right] \right) \right].$$

Now if we use the control \mathbf{u}^* as in (2.3) then we get an equality in the above in place of inequality. The existence of an \mathbf{u}^* satisfying (2.3) is ensured by a standard measurable selection theorem [2]. Hence the theorem follows. \square

Next we prove that there exists a smooth, bounded solution to (2.2).

Theorem 2.2. *Assume (A1)-(A3). Then there exists a unique solution to (2.2) in $C_b([0, T] \times S) \cap C^1([0, T] \times S)$.*

Proof. Let $\varphi(t, i) = e^{-\gamma_0 t} \psi(t, i)$. Substituting in (2.2) we get

$$\begin{cases} \frac{d\psi}{dt} - \gamma_0 \psi + \inf_{u \in U_i} [\theta c(t, i, u) \psi(t, i) + \sum_j \lambda_{ij}(u) \psi(t, j)] = 0 \text{ on } [0, T] \times S \\ \psi(T, i) = e^{\gamma_0 T} e^{\theta g(i)}. \end{cases} \quad (2.4)$$

Consider the following integral equation:

$$\psi(t, i) = e^{\gamma_0 t} e^{\theta g(i)} + e^{\gamma_0 t} \int_t^T e^{-\gamma_0 s} \inf_{u \in U_i} [\theta c(s, i, u) \psi(s, i) + \sum_j \lambda_{ij}(u) \psi(s, j)] ds. \quad (2.5)$$

Define $\mathcal{T} : C_b([0, T] \times S) \rightarrow C_b([0, T] \times S)$ by

$$\mathcal{T}\psi(t, i) = e^{\gamma_0 t} e^{\theta g(i)} + e^{\gamma_0 t} \int_t^T e^{-\gamma_0 s} \inf_{u \in U_i} [\theta c(s, i, u) \psi(s, i) + \sum_j \lambda_{ij}(u) \psi(s, j)] ds.$$

Then

$$\begin{aligned}
|\mathcal{T}\psi_1(t, i) - \mathcal{T}\psi_2(t, i)| &\leq e^{\gamma_0 t} \int_t^T e^{-\gamma_0 s} \{ \theta \|c\| \|\psi_1 - \psi_2\| + 2M \|\psi_1 - \psi_2\| \} ds \\
&= (2M + \theta \|c\|) \|\psi_1 - \psi_2\| e^{\gamma_0 t} \int_t^T e^{-\gamma_0 s} ds \\
&= \frac{2M + \theta \|c\|}{\gamma_0} \|\psi_1 - \psi_2\| e^{\gamma_0 t} (e^{-\gamma_0 t} - e^{-\gamma_0 T}) \\
&\leq \frac{2M + \theta \|c\|}{\gamma_0} \|\psi_1 - \psi_2\|,
\end{aligned}$$

where M is as in **(A2)**. Hence for $\gamma_0 = 2M + \theta \|c\| + 1$, \mathcal{T} is a contraction and thus by Banach's fixed point theorem there exists a unique solution to (2.5) in $C_b([0, T] \times S)$. Using **(A1)**-**(A3)**, the boundedness and continuity of the cost function c , it follows that ψ is in $C_b([0, T] \times S) \cap C^1[0, T] \times S$. Then it follows that $\varphi(t, i) = e^{-\gamma_0 t} \psi(t, i)$ is a solution of (2.2). The uniqueness follows from the previous theorem. \square

Thus combining the above two theorems we get the following theorem:

Theorem 2.3. *Under **(A1)**-**(A3)**, the value function V_T is the unique solution to (2.2) in $C_b([0, T] \times S) \cap C^1([0, T] \times S)$. An optimal control is given by the Markov control $U_t^* = \mathbf{u}^*(t, X_{t-})$ where \mathbf{u}^* satisfies*

$$\begin{aligned}
&\inf_u [\theta c(t, i, u) V_T(t, i) + \sum_j \lambda_{ij}(u) V_T(t, j)] \\
&= [\theta c(t, i, \mathbf{u}^*(t, i)) V_T(t, i) + \sum_j \lambda_{ij}(\mathbf{u}^*(t, i)) V_T(t, j)].
\end{aligned} \tag{2.6}$$

Now since logarithm is an increasing function the following theorem is now evident.

Theorem 2.4. *Let φ be the unique solution of (2.2) in $C_b([0, T] \times S) \cap C^1([0, T] \times S)$. Define $\psi = \theta^{-1} \log \varphi$. Then*

$$\psi(t, i) = \inf_{\mathcal{U}} \frac{1}{\theta} \log \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \left[\int_0^T c(s, X_s, U_s) ds + g(X_T) \right] \right) \right].$$

Moreover the Markov control given by (2.6) is again an optimal control in this case.

3. Discounted Cost Case

In this section we turn our attention towards infinite horizon discounted cost problem. Define

$$V_\alpha(\theta, i) = \inf_{\mathcal{U}} I_\alpha(\theta, i, \mathbf{u}) \tag{3.1}$$

where $I_\alpha(\theta, i, \mathbf{u})$ is as in (1.5). The function V_α is called the α -discounted value function. Our aim is to characterise the value function and to obtain an optimal control.

Instead of working with V_α we first start with

$$W_\alpha(\theta, i) = \inf_{\mathcal{U}} \exp [\theta I_\alpha(\theta, i, \mathbf{u})]. \tag{3.2}$$

Formally, for any $T > 0$

$$\begin{aligned} W_\alpha(\theta, i) &= \inf_{\mathcal{U}} \mathbb{E}_i^{\mathbf{u}} \left\{ \exp \left[\theta \int_0^T e^{-\alpha t} c(X_t, U_t) dt + \theta \int_T^\infty e^{-\alpha t} c(X_t, U_t) dt \right] \right\} \\ &= \inf_{\mathcal{U}} \mathbb{E}_i^{\mathbf{u}} \left\{ \exp \left[\theta \int_0^T e^{-\alpha t} c(X_t, U_t) dt \right] \mathbb{E}_{X_T}^{\mathbf{u}} \left(\exp \left[\theta e^{-\alpha T} \int_0^\infty e^{-\alpha t} c(X_t, U_t) dt \right] \right) \right\} \\ &= \inf_{\mathcal{U}} \mathbb{E}_i^{\mathbf{u}} \left\{ \exp \left[\theta \int_0^T e^{-\alpha t} c(X_t, U_t) dt \right] W_\alpha(\theta e^{-\alpha T}, X_T) \right\}. \end{aligned}$$

If $W_\alpha(\cdot, i)$ is smooth then, using Dynkin's formula and some heuristic arguments we obtain that W_α should satisfy

$$\begin{cases} \alpha \theta \frac{dW_\alpha}{d\theta}(\theta, i) = \inf_{u \in U_i} \left[\theta c(i, u) W_\alpha(\theta, i) + \sum_j \lambda_{ij}(u) W_\alpha(\theta, j) \right] \\ \text{with } \lim_{\theta \rightarrow 0} W_\alpha(\theta, i) = 1. \end{cases} \quad (3.3)$$

Equation (3.3) is known as the HJB equation for the cost criterion (3.2). Now starting with (3.3) the following verification theorem can be obtained.

Theorem 3.1. *Assume that there exists a bounded, smooth (continuously differentiable in the first variable) function $w(\theta, i)$ such that*

$$\alpha \theta \frac{dw}{d\theta}(\theta, i) = \inf_{u \in U_i} \left[\theta c(i, u) w(\theta, i) + \sum_j \lambda_{ij}(u) w(\theta, j) \right] \text{ on } (0, 1) \times S \quad (3.4)$$

and $w(\theta, i) \rightarrow 1$ as $\theta \rightarrow 0$ uniformly in i . Then $w(\theta, i) = W_\alpha(\theta, i)$. Furthermore an optimal control for the cost criterion is given by (3.2) is given by

$$U_t^* = \mathbf{u}^*(\theta e^{-\alpha t}, X_{t-}) \quad (3.5)$$

where \mathbf{u}^* is given by

$$\begin{aligned} \inf_{u \in U_i} [\theta c(i, u) w(\theta, i) + \sum_j \lambda_{ij}(u) w(\theta, j)] = \\ [\theta c(i, \mathbf{u}^*(\theta, i)) w(\theta, i) + \sum_j \lambda_{ij}(\mathbf{u}^*(\theta, i)) w(\theta, j)]. \end{aligned} \quad (3.6)$$

Proof. Define $\theta_t = \theta e^{-\alpha t}$ and

$$\Psi_t = \exp \left\{ \int_0^t \theta_s c(X_s, U_s) ds \right\}$$

for any arbitrary Markov control $U_t = \mathbf{u}(t, X_{t-})$. Then by Feynman - Kac formula we get

$$\begin{aligned} \mathbb{E}_i^{\mathbf{u}} \{ \Psi_T w(\theta_T, X_T) \} - w(\theta, i) \\ = \mathbb{E}_i^{\mathbf{u}} \left\{ \int_0^T \Psi_t \left[-\alpha \theta_t \frac{dw}{d\theta}(\theta_t, X_t) + \theta_t c(X_t, U_t) w(\theta_t, X_t) + \sum_j \lambda_{X_t j}(U_t) w(\theta_t, X_t) \right] dt \right\}. \end{aligned}$$

Since w satisfies (3.4), the term on the righthand side above is non-negative. Therefore we get

$$w(\theta, i) \leq \mathbb{E}_i^{\mathbf{u}} \{ \Psi_T w(\theta_T, X_T) \}.$$

Now $\theta_T \rightarrow 0$ as $T \rightarrow \infty$ and hence $w(\theta_T, X_T) \rightarrow 1$. Thus we get

$$w(\theta, i) \leq \mathbb{E}_i^u \left\{ \exp \left[\theta \int_0^\infty e^{-\alpha t} c(X_t, U_t) dt \right] \right\}. \quad (3.7)$$

Similarly if we take the Markov control U^* given by (3.5) and (3.6) then we get equality in (3.7) in place of inequality. Hence the theorem follows. \square

The following result is now evident.

Corollary 3.2. *For w as in Theorem 3.1, define $v(\theta, i) = \theta^{-1} \log w(\theta, i)$. Then $v = V_\alpha$, where V_α is as in (3.1) .*

Now we prove that the HJB equation (3.4) indeed has a smooth solution. To this end we first prove the following result.

Proposition 3.3. *Let $\epsilon > 0$ be arbitrary but fixed. There exists a function W_ϵ in $C_b([\epsilon, 1] \times S) \cap C^1((\epsilon, 1) \times S)$ such that W_ϵ satisfies*

$$\begin{cases} \alpha \theta \frac{dW_\epsilon}{d\theta}(\theta, i) = \inf_{u \in U_i} \left[\theta c(i, u) W_\epsilon(\theta, i) + \sum_j \lambda_{ij}(u) W_\epsilon(\theta, j) \right] \text{ on } (\epsilon, 1) \times S \\ W_\epsilon(\epsilon, i) = e^{\frac{\epsilon}{\alpha} \|c\|} := h_\epsilon(i). \end{cases} \quad (3.8)$$

Proof. Let $\delta > 0$ which will be specified soon. Define $T : C_b([\epsilon, \epsilon + \delta] \times S) \rightarrow C_b([\epsilon, \epsilon + \delta] \times S)$ by

$$Tf(\eta, i) = e^{\frac{\epsilon}{\alpha} \|c\|} + \frac{1}{\alpha} \int_\epsilon^\eta \inf_{u \in U_i} [c(i, u) f(\theta, i) + \frac{1}{\theta} \sum_j \lambda_{ij}(u) f(\theta, j)] d\theta.$$

Then

$$|Tf_1(\eta, i) - Tf_2(\eta, i)| \leq \frac{1}{\alpha} \left[\|c\| \delta \|f_1 - f_2\| + \frac{2M}{\epsilon} \delta \|f_1 - f_2\| \right],$$

where M is as in **(A2)**. Choose δ such that

$$\beta := \frac{1}{\alpha} \left[\|c\| \delta + \frac{2M}{\epsilon} \delta \right]$$

is strictly less than 1. Then T is a contraction. Hence by Banach's fixed point theorem there exists a W in $C_b([\epsilon, \epsilon + \delta] \times S)$ which is the unique fixed point of T . Now assumptions **(A1)**-**(A3)** and the continuity of c imply that W is in $C_b([\epsilon, \epsilon + \delta] \times S) \cap C^1((\epsilon, \epsilon + \delta] \times S)$. Thus it follows that W satisfies (3.8) on $[\epsilon, \epsilon + \delta] \times S$. Proceeding in this way we will get a function $W_\epsilon \in C_b([\epsilon, 1] \times S) \cap C^1((\epsilon, 1) \times S)$ which satisfies (3.8). \square

Next we take limit $\epsilon \rightarrow 0$ of W_ϵ and show that the limit satisfies (3.4). In particular we prove the following theorem:

Theorem 3.4. *Assume **(A1)**-**(A3)** and further assume that S is finite. Then there exists a unique solution W in the class $C_b((0, 1) \times S) \cap C^1((0, 1) \times S)$ to the equation*

$$\begin{cases} \alpha \theta \frac{dW}{d\theta}(\theta, i) = \inf_{u \in U_i} \left[\theta c(i, u) W(\theta, i) + \sum_j \lambda_{ij}(u) W(\theta, j) \right] \text{ on } (0, 1) \times S \\ \text{with } \lim_{\theta \rightarrow 0} W(\theta, i) = 1. \end{cases}$$

Proof. Using Dynkin's formula it can be shown that W_ϵ has the following stochastic representation:

$$W_\epsilon(\theta, i) = \inf_{\mathcal{U}} \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left(\theta \int_0^{T_\epsilon} e^{-\alpha t} c(X_t, U_t) dt \right) \right]$$

where h_ϵ is as in (5.2) and $T_\epsilon = \inf\{t \geq 0 : \theta_t = \epsilon\}$, i.e., $T_\epsilon = \frac{\log(\frac{\theta}{\epsilon})}{\alpha}$. From this representation of W_ϵ we can deduce that for every $\epsilon > 0$

$$0 \leq W_\epsilon(\theta, i) \leq e^{\frac{\theta}{\alpha} \|c\|} \leq e^{\frac{\|c\|}{\alpha}}.$$

Now we show that $\frac{dW_\epsilon}{d\theta}$ is also uniformly (in $\epsilon > 0$) bounded. For any arbitrary Markov control \mathbf{u} ,

$$\begin{aligned} & \left| \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left((\theta + \delta) \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) \right] - \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left(\theta \int_0^{T_\epsilon} e^{-\alpha t} c(X_t, U_t) dt \right) \right] \right| \\ & \leq I_1 + I_2 \end{aligned}$$

where $(\theta + \delta)e^{-\alpha T_\epsilon^\delta} = \epsilon$ and

$$I_1 = \left| \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left((\theta + \delta) \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) \right] - \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left(\theta \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) \right] \right|,$$

$$I_2 = \left| \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left(\theta \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) \right] - \mathbb{E}_i^{\mathbf{u}} \left[h_\epsilon \exp \left(\theta \int_0^{T_\epsilon} e^{-\alpha t} c(X_t, U_t) dt \right) \right] \right|.$$

Now

$$\begin{aligned} I_1 & \leq e^{\frac{\|c\|}{\alpha}} \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) \left| \exp \left(\delta \int_0^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) - 1 \right| \right] \\ & \leq C_1 e^{\frac{2\|c\|}{\alpha}} \delta \frac{\|c\|}{\alpha} \end{aligned}$$

for some constant $C_1 > 0$ and for $\delta > 0$ small enough. Similarly for I_2 we have

$$\begin{aligned} I_2 & \leq e^{\frac{\|c\|}{\alpha}} \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^{T_\epsilon} e^{-\alpha t} c(X_t, U_t) dt \right) \left| \exp \left(\theta \int_{T_\epsilon}^{T_\epsilon^\delta} e^{-\alpha t} c(X_t, U_t) dt \right) - 1 \right| \right] \\ & \leq e^{\frac{2\|c\|}{\alpha}} \left[e^{\frac{\theta\|c\|}{\alpha}} (e^{-\alpha T_\epsilon} - e^{-\alpha T_\epsilon^\delta}) - 1 \right]. \end{aligned}$$

But $\theta e^{-\alpha T_\epsilon} - \theta e^{-\alpha T_\epsilon^\delta} = \delta e^{-\alpha T_\epsilon^\delta} = \frac{\epsilon \delta}{\theta + \delta}$. Hence we have

$$\begin{aligned} I_2 & \leq e^{\frac{2\|c\|}{\alpha}} \left[e^{\frac{\theta\|c\|}{\alpha} \frac{\epsilon \delta}{\theta + \delta}} - 1 \right] \\ & \leq C_2 e^{\frac{2\|c\|}{\alpha}} \delta \frac{\|c\|}{\alpha} \end{aligned}$$

for some constant $C_2 > 0$ and for $\delta > 0$ small enough. Hence we can conclude that for $\delta > 0$, small enough

$$|W_\epsilon(\theta + \delta, i) - W_\epsilon(\theta, i)| \leq C_3 e^{\frac{2\|c\|}{\alpha}} \delta \frac{\|c\|}{\alpha}$$

for some constant $C_3 > 0$.

Similarly for $\delta < 0$, small enough, we can get an estimate of the type

$$|W_\epsilon(\theta + \delta, i) - W_\epsilon(\theta, i)| \leq C_3 e^{\frac{2\|c\|}{\alpha}} |\delta| \frac{\|c\|}{\alpha}.$$

Thus we get that $\frac{dW_\epsilon}{d\theta}$ is uniformly bounded in $\epsilon > 0$.

Now define

$$\widetilde{W}_\epsilon(\theta, i) = \begin{cases} W_\epsilon(\theta, i) & \text{for } \theta > \epsilon \\ h_\epsilon(i) & \text{for } \theta \leq \epsilon. \end{cases}$$

Then \widetilde{W}_ϵ satisfies the same bounds. Now since \widetilde{W}_ϵ is uniformly bounded and $\frac{d\widetilde{W}_\epsilon}{d\theta}$ is also uniformly bounded, by Ascoli - Arzela theorem there exists a function W in $C_b((0, 1) \times S)$ and a sequence $\epsilon_n \rightarrow 0$ such that $\widetilde{W}_{\epsilon_n} \rightarrow W$ uniformly over compact subsets of $(0, 1) \times S$. Also by the definition of \widetilde{W}_ϵ , $W(\theta, i) \rightarrow 1$ as $\theta \rightarrow 0$. Now taking $\varphi \in C_c^\infty(0, 1)$ we get

$$\begin{aligned} & - \int_0^1 \alpha \widetilde{W}_{\epsilon_n} \frac{d(\theta\varphi)}{d\theta} d\theta = \int_0^1 \alpha \theta \frac{d\widetilde{W}_{\epsilon_n}}{d\theta} \varphi(\theta) d\theta \\ & = \int_0^1 \inf_{u \in U_i} \left[\theta c(i, u) \widetilde{W}_{\epsilon_n}(\theta, i) + \sum_j \lambda_{ij}(u) \widetilde{W}_{\epsilon_n}(\theta, j) \right] \varphi(\theta) d\theta \\ & - \int_0^{\epsilon_n} \inf_{u \in U_i} \left[\theta c(i, u) \widetilde{W}_{\epsilon_n}(\theta, i) + \sum_j \lambda_{ij}(u) \widetilde{W}_{\epsilon_n}(\theta, j) \right] \varphi(\theta) d\theta. \end{aligned}$$

Now taking limit $n \rightarrow \infty$ we get

$$- \int_0^1 \alpha W \frac{d(\theta\varphi)}{d\theta} d\theta = \int_0^1 \inf_{u \in U_i} \left[\theta c(i, u) W(\theta, i) + \sum_j \lambda_{ij}(u) W(\theta, j) \right] \varphi(\theta) d\theta.$$

Thus

$$\alpha \theta \frac{dW}{d\theta} = \inf_{u \in U_i} \left[\theta c(i, u) W(\theta, i) + \sum_j \lambda_{ij}(u) W(\theta, j) \right]$$

in the sense of distribution. But by our assumptions the righthand side is a continuous function. Therefore $\frac{dW}{d\theta}$ is in $C((0, 1) \times S)$. Thus W is a smooth solution to the HJB equation (5.2). The uniqueness follows from Theorem 3.1. \square

This immediately yields the following result:

Theorem 3.5. *Assume (A1)-(A3) and that S is finite. Then the value function V_α as in (3.1) is the unique solution in $C_b((0, 1) \times S) \cap C^1((0, 1) \times S)$ to*

$$\alpha \theta \left[v + \theta \frac{dv}{d\theta} \right] e^{\theta v} = \inf_{u \in U_i} \left[\theta c e^{\theta v} + \sum_j \lambda_{ij}(u) e^{\theta v(\theta, j)} \right] \text{ on } (0, 1) \times S$$

$$\text{with } \lim_{\theta \rightarrow 0} v(\theta, i) = \inf_{\mathcal{U}} \mathbb{E}_i^{\mathbf{u}} \int_0^\infty e^{-\alpha t} c(X_t, U_t) dt.$$

An optimal control is given by the Markov control $U_t = \mathbf{u}^*(\theta e^{-\alpha t}, X_{t-})$ where \mathbf{u}^* is given by

$$\begin{aligned} & \inf_{u \in U_i} \left[\theta c(i, u) e^{\theta V_\alpha} + \sum_j \lambda_{ij}(u) e^{\theta V_\alpha(\theta, j)} \right] = \\ & \left[\theta c(i, \mathbf{u}^*(\theta, i)) e^{\theta V_\alpha} + \sum_j \lambda_{ij}(\mathbf{u}^*(\theta, i)) e^{\theta V_\alpha(\theta, j)} \right]. \end{aligned}$$

Remark 3.6. *The finiteness of the state space in Theorem 3.4 is forced upon by the uniformity in the boundary condition in 3.1. Note that the limiting procedure that we have employed only yields that $\lim_{\theta \rightarrow 0} W(\theta, i) = 1$ for each i . Hence the finiteness assumption on S . Note that a similar situation arises in the risk-sensitive control of diffusion processes [22]. In [22] the authors treat periodic diffusions for which the state space is a torus which is compact.*

4. Infinite Horizon Average Cost

In this section we study the infinite horizon average cost case. In order to study the average cost case we make some further assumptions on our model.

(A4) For every stationary control \mathbf{u} , the corresponding Markov chain is irreducible.

(A5) There exists a Lyapunov function $V : S \rightarrow \mathbb{R}^+$, an unbounded function $W : S \rightarrow [1, \infty)$ and constants $\delta > 0$ and $b < \infty$ such that

$$e^{-V(i)} \sum_j \lambda_{ij}(u) e^{V(j)} \leq -\delta W(i) + b 1_{\{0\}}(i) \text{ for all } i, u. \quad (4.1)$$

An important consequence of **(A5)** is the following lemma:

Lemma 4.1. *Let $\eta < \delta$ and*

$$\tau_0 = \inf\{t > 0 : X_t = 0\}. \quad (4.2)$$

Then

$$\sup_{\mathbf{u}} \mathbb{E}_i^{\mathbf{u}} e^{\eta \tau_0} \leq e^{V(i)}.$$

Proof. Let

$$\tau_n = \sup\{t \geq 0 : X_t \leq n\}.$$

If $X_0 \geq n$, then τ_n is 0. Assumption **(A5)** implies that there exists a \tilde{b} and a $\tilde{V} = e^V$ such that

$$\sum_j \lambda_{ij}(u) \tilde{V}(j) \leq -\delta \tilde{V}(i) + \tilde{b} 1_{\{0\}}(i).$$

By Dynkin's formula we get

$$\begin{aligned} \mathbb{E}_i^{\mathbf{u}} e^{\eta(\tau_0 \wedge \tau_n \wedge n)} \tilde{V}(X_{\tau_0 \wedge \tau_n \wedge n}) &= \tilde{V}(i) + \mathbb{E}_i^{\mathbf{u}} \int_0^{\tau_0 \wedge \tau_n \wedge n} e^{\eta s} (\Lambda^{\mathbf{u}} + \eta) \tilde{V}(X_s) ds \\ &\leq \tilde{V}(i) + \mathbb{E}_i^{\mathbf{u}} \int_0^{\tau_0 \wedge \tau_n \wedge n} e^{\eta s} (\eta - \delta) \tilde{V}(X_s) ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \tilde{V}(i) &\geq \mathbb{E}_i^{\mathbf{u}} e^{\eta(\tau_0 \wedge \tau_n \wedge n)} \tilde{V}(X_{\tau_0 \wedge \tau_n \wedge n}) + \mathbb{E}_i^{\mathbf{u}} \int_0^{\tau_0 \wedge \tau_n \wedge n} e^{\eta s} (-\eta + \delta) \tilde{V}(X_s) ds \\ &\geq \mathbb{E}_i^{\mathbf{u}} e^{\eta(\tau_0 \wedge \tau_n \wedge n)}. \end{aligned}$$

Now letting $n \rightarrow \infty$ we get the desired result. \square

Finally we make the following assumption

(A6) For τ_0 as defined in (4.2), $\sup_{i, \mathbf{u}} \mathbb{E}_i^{\mathbf{u}} \tau_0 < \infty$.

Remark 4.2. *If the state space is finite then it can be easily seen that (A5) implies (A6).*

Now we state and prove the main theorem of this section:

Theorem 4.3. *Under (A1)-(A6), an optimal control for the risk-sensitive average cost criterion exists for θ and c satisfying $\theta \|c\| < \delta$ where δ is as in (4.1).*

Proof. Let

$$\theta \rho^{\mathbf{u}} = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^T c(X_t, \mathbf{u}(X_t)) dt \right) \right]. \quad (4.3)$$

The existence of the above limit follows from the multiplicative ergodic theorems proved in [18] and [19]. Moreover it also follows from the results in [18] and [19] that the limit in (4.3) is the principal eigenvalue for the operator $\Lambda^{\mathbf{u}} + \theta c$ and has a positive eigenfunction which belongs to the class L_V^∞ , i.e., if we denote an eigenfunction by $h^{\mathbf{u}}$ then $\sup \frac{|h^{\mathbf{u}}(i)|}{V(i)} < \infty$. Thus the following equation holds

$$\sum_j \lambda_{ij}(\mathbf{u}(i)) h^{\mathbf{u}}(j) + \theta c(i, \mathbf{u}(i)) h^{\mathbf{u}}(i) = \rho^{\mathbf{u}} \theta h^{\mathbf{u}}(i). \quad (4.4)$$

Equation (4.4) is referred to as the Poisson equation. Now it is clear that if $h^{\mathbf{u}}$ satisfies (4.4) then so does any scalar multiple of $h^{\mathbf{u}}$. Therefore without any loss of generality we may assume that $h^{\mathbf{u}}(0) = 1$. With this restriction, using Dynkin's formula and the fact that $h^{\mathbf{u}}$ satisfies (4.4) we get the following stochastic representation for $h^{\mathbf{u}}$:

$$h^{\mathbf{u}}(i) = \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^{\tau_0} (c(X_s, \mathbf{u}(X_s)) - \rho^{\mathbf{u}}) ds \right) \right]. \quad (4.5)$$

Now using the stochastic representation of $h^{\mathbf{u}}$ we derive bounds on $h^{\mathbf{u}}$. First we derive an upper bound. We have

$$\begin{aligned} h^{\mathbf{u}}(i) &= \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^{\tau_0} (c(X_s, \mathbf{u}(X_s)) - \rho^{\mathbf{u}}) ds \right) \right] \\ &\leq \mathbb{E}_i^{\mathbf{u}} e^{\theta \|c\| \tau_0} \\ &\leq e^{V(i)} \end{aligned}$$

by Lemma 4.1, provided $\theta \|c\| < \delta$.

This upper bound shows that bound on $h^{\mathbf{u}}$ is uniform in \mathbf{u} . Next we obtain a lower bound.

We have

$$\begin{aligned}
h^{\mathbf{u}}(i) &= \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^{\tau_0} (c(X_s, \mathbf{u}(X_s)) - \rho^{\mathbf{u}}) ds \right) \right] \\
&\geq \exp \left\{ \mathbb{E}_i^{\mathbf{u}} \theta \int_0^{\tau_0} (c(X_s, \mathbf{u}(X_s)) - \rho^{\mathbf{u}}) ds \right\} \\
&\geq \exp(-\theta \rho^{\mathbf{u}} \mathbb{E}_i^{\mathbf{u}} \tau_0) \\
&\geq \exp(-\theta \|c\| \mathbb{E}_i^{\mathbf{u}} \tau_0) \\
&> \epsilon
\end{aligned}$$

for some $\epsilon > 0$. In the above sequence of inequalities the second one follows from Jensen's inequality and the last one follows from **(A6)**.

Let

$$\rho^* = \inf_{\mathbf{u}} \rho^{\mathbf{u}}. \quad (4.6)$$

Next we show that there exists a control \mathbf{u}^* which attains the infimum in (4.6). From (4.6) it follows that there exists a sequence \mathbf{u}_n such that $\rho^{\mathbf{u}_n} \rightarrow \rho^*$. Since each U_i is compact there exists a subsequence which is also denoted by \mathbf{u}_n again and a \mathbf{u}^* such that

$$\mathbf{u}_n \rightarrow \mathbf{u}^* \text{ pointwise.}$$

Again since $h^{\mathbf{u}_n}$ is pointwise bounded, there exists a subsequence which we call $h^{\mathbf{u}_n}$ again such that

$$h^{\mathbf{u}_n}(i) \rightarrow h^*(i) \text{ for each } i,$$

for some h^* and $\inf_i h^*(i) \geq \epsilon$. Therefore by using Fatou's lemma we have

$$\begin{aligned}
\sum_{j \neq i} \lambda_{ij}(\mathbf{u}^*(i)) h^*(j) &\leq \liminf_{n \rightarrow \infty} \sum_{j \neq i} \lambda_{ij}(\mathbf{u}_n(i)) h^{\mathbf{u}_n}(j) \\
&= \liminf_{n \rightarrow \infty} [-\lambda_{ii}(\mathbf{u}_n(i)) h^{\mathbf{u}_n}(i) - \theta c(i, \mathbf{u}_n(i)) h^{\mathbf{u}_n}(i) + \theta \rho^{\mathbf{u}_n} h^{\mathbf{u}_n}(i)] \\
&= -\lambda_{ii}(\mathbf{u}^*(i)) h^*(i) - \theta c(i, \mathbf{u}^*(i)) h^*(i) + \theta \rho^* h^*(i).
\end{aligned}$$

Thus we get

$$\sum_j \lambda_{ij}(\mathbf{u}^*(i)) h^*(j) + \theta c(i, \mathbf{u}^*(i)) h^*(i) \leq \theta \rho^* h^*(i).$$

Now we claim that

$$\rho^* = \rho^{\mathbf{u}^*}.$$

Indeed, with τ_n as in the proof of Lemma 4.1 we get from Dynkin's formula

$$\begin{aligned} & \mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^{T \wedge \tau_n} c(X_s, \mathbf{u}^*(X_s)) ds \right) h^*(X_{T \wedge \tau_n}) \right] \\ &= h^*(i) + \mathbb{E}_i^{\mathbf{u}^*} \left[\int_0^{T \wedge \tau_n} \exp \left(\theta \int_0^t c(X_s, \mathbf{u}^*(X_s)) ds \right) (\Lambda^{\mathbf{u}^*} + \theta c) h^*(X_t) dt \right] \\ &\leq h^*(i) + \theta \rho^* \mathbb{E}_i^{\mathbf{u}^*} \left[\int_0^{T \wedge \tau_n} \exp \left(\theta \int_0^t c(X_s, \mathbf{u}^*(X_s)) ds \right) h^*(X_t) dt \right] \\ &\leq h^*(i) + \theta \rho^* \int_0^T \mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^{t \wedge \tau_n} c(X_s, \mathbf{u}^*(X_s)) ds \right) h^*(X_{t \wedge \tau_n}) \right] dt. \end{aligned}$$

Hence by Gronwall's inequality we have

$$\mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^{T \wedge \tau_n} c(X_s, \mathbf{u}^*(X_s)) ds \right) h^*(X_{T \wedge \tau_n}) \right] \leq h^*(i) e^{\theta \rho^* T}.$$

Therefore letting $n \rightarrow \infty$ we have

$$h^*(i) e^{\theta \rho^* T} \geq \mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^T c(X_s, \mathbf{u}^*(X_s)) ds \right) \right].$$

which implies

$$\theta \rho^* \geq \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^T c(X_s, \mathbf{u}^*(X_s)) ds \right) \right] = \theta \rho^{\mathbf{u}^*}.$$

Hence $\rho^* = \rho^{\mathbf{u}^*}$. □

5. Policy Improvement Algorithm

In the previous section we have proved the existence of an optimal control. But our theorem is purely existential and does not give an algorithm to find an optimal control. In this section we focus on the computational approach for finding an optimal stationary control. Since we are concerned with algorithm in this section we assume that both the state and action spaces are finite. Now we describe the policy improvement algorithm.

Algorithm

Step 1: Start with any initial stationary control \mathbf{u}_1 . For this \mathbf{u}_1

$$\rho^{\mathbf{u}_1} = \lim_{T \rightarrow \infty} \frac{1}{\theta T} \log \mathbb{E}_i^{\mathbf{u}_1} \left[\exp \left(\theta \int_0^T c(X_t, \mathbf{u}_1(X_t)) dt \right) \right]$$

and

$$h^{\mathbf{u}_1}(i) = \mathbb{E}_i^{\mathbf{u}_1} \left[\exp \left(\theta \int_0^{\tau_0} (c(X_t, \mathbf{u}_1(X_t)) - \rho^{\mathbf{u}_1}) dt \right) \right].$$

We know from previous section that $h^{\mathbf{u}_1}$ satisfies the Poisson equation

$$\sum_j \lambda_{ij}(\mathbf{u}_1(i)) h^{\mathbf{u}_1}(j) + \theta c(i, \mathbf{u}_1(i)) h^{\mathbf{u}_1}(i) = \theta \rho^{\mathbf{u}_1} h^{\mathbf{u}_1}(i)$$

satisfying the constraint $h^{\mathbf{u}_1}(0) = 1$.

Step 2: Define \mathbf{u}_2 to be the stationary control which minimizes

$$\min_{u \in U_i} \left[\theta c(i, u) h^{\mathbf{u}_1}(i) + \sum_j \lambda_{ij}(u) h^{\mathbf{u}_1}(j) \right].$$

With $\rho^{\mathbf{u}_2}$ and $h^{\mathbf{u}_2}$ as above continue the procedure.

Theorem 5.1. *The above algorithm leads to an optimal control in finite number of steps.*

Proof. In order to prove that this algorithm comes up with an optimal control in a finite number of steps we first claim that

$$\rho^{\mathbf{u}_{n+1}} \leq \rho^{\mathbf{u}_n}. \quad (5.1)$$

Indeed, from the definition of \mathbf{u}_{n+1} we have

$$\begin{aligned} \sum_j \lambda_{ij}(\mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(i) &\leq \sum_j \lambda_{ij}(\mathbf{u}_n(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_n(i)) h^{\mathbf{u}_n}(i) \\ &= \theta \rho^{\mathbf{u}_n} h^{\mathbf{u}_n}(i). \end{aligned}$$

Now using arguments involving Dynkin's formula as in the previous section it can be proved that $\rho^{\mathbf{u}_{n+1}} \leq \rho^{\mathbf{u}_n}$.

Our second claim is that, suppose for some n and for all i

$$\sum_j \lambda_{ij}(\mathbf{u}_n(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_n(i)) h^{\mathbf{u}_n}(i) = \sum_j \lambda_{ij}(\mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(i) \quad (5.2)$$

then \mathbf{u}_n is optimal.

To prove this observe that if \mathbf{u}_n is as in (5.2) then

$$\begin{aligned} \theta \rho^{\mathbf{u}_n} h^{\mathbf{u}_n}(i) &= \sum_j \lambda_{ij}(\mathbf{u}_n(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_n(i)) h^{\mathbf{u}_n}(i) \\ &= \sum_j \lambda_{ij}(\mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(j) + \theta c(i, \mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(i) \\ &= \min_{u \in U_i} [\theta c(i, u) h^{\mathbf{u}_n}(i) + \sum_j \lambda_{ij}(u) h^{\mathbf{u}_n}(j)]. \end{aligned} \quad (5.3)$$

Now for any stationary control \mathbf{u} we have by Dynkin's formula

$$\begin{aligned} \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^T (c(X_t, \mathbf{u}(X_t)) - \rho^{\mathbf{u}_n}) dt \right) h^{\mathbf{u}_n}(X_T) \right] \\ = h^{\mathbf{u}_n}(i) + \mathbb{E}_i^{\mathbf{u}} \int_0^T \exp \left(\theta \int_0^t (c(X_s, \mathbf{u}(X_s)) - \rho^{\mathbf{u}_n}) ds \right) [\Lambda^{\mathbf{u}} + \theta c - \theta \rho^{\mathbf{u}_n}] h^{\mathbf{u}_n}(X_t) dt \\ \geq h^{\mathbf{u}_n}(i). \end{aligned}$$

The last inequality follows from (5.3). Thus we have

$$h^{\mathbf{u}_n}(i) \leq \max_i h^{\mathbf{u}_n}(i) \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^T (c(X_t, \mathbf{u}(X_t)) - \rho^{\mathbf{u}_n}) dt \right) \right].$$

This implies that

$$\rho^{\mathbf{u}_n} \leq \lim_{T \rightarrow \infty} \frac{1}{\theta T} \log \mathbb{E}_i^{\mathbf{u}} \left[\exp \left(\theta \int_0^T c(X_t, \mathbf{u}(X_t)) dt \right) \right] = \rho^{\mathbf{u}}.$$

Hence the claim.

Our final claim is that if \mathbf{u}_n is not an optimal control then the inequality in (5.1) is actually

a strict inequality.

Since \mathbf{u}_n is not optimal, we have

$$\sum_j \lambda_{ij}(\mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(i) + \theta c(i, \mathbf{u}_{n+1}(i)) h^{\mathbf{u}_n}(i) - \theta \rho^{\mathbf{u}_n} h^{\mathbf{u}_n}(i) = -g(i),$$

where g is a non-negative function and there exists at least one i such that $g(i) > 0$.

Therefore for any $T > 0$ it follows from Dynkin's formula that

$$\begin{aligned} & \mathbb{E}_i^{\mathbf{u}_{n+1}} \left[\exp \left(\theta \int_0^T (c(X_s, \mathbf{u}_{n+1}(X_s)) - \rho^{\mathbf{u}_n}) ds \right) h^{\mathbf{u}_n}(X_T) \right] \\ &= h^{\mathbf{u}_n}(i) + \mathbb{E}_i^{\mathbf{u}_{n+1}} \left[\int_0^T \exp \left(\theta \int_0^t (c(X_s, \mathbf{u}_{n+1}(X_s)) - \rho^{\mathbf{u}_n}) ds \right) \left(\theta c_{n+1} + \Lambda^{n+1} - \theta \rho^{\mathbf{u}_n} \right) h^{\mathbf{u}_n}(X_t) dt \right] \\ &= h^{\mathbf{u}_n}(i) - \mathbb{E}_i^{\mathbf{u}_{n+1}} \left[\int_0^T \exp \left(\theta \int_0^t (c(X_s, \mathbf{u}_{n+1}(X_s)) - \rho^{\mathbf{u}_n}) ds \right) g(X_t) dt \right] \\ &= h^{\mathbf{u}_n}(i) - \int_0^T \exp(-(\rho^{\mathbf{u}_n} - \rho^{\mathbf{u}_{n+1}})t) \exp \left(\theta \int_0^t (c(X_s, \mathbf{u}_{n+1}(X_s)) - \rho^{\mathbf{u}_{n+1}}) ds \right) g(X_t) dt \\ &= h^{\mathbf{u}_n}(i) - h^{\mathbf{u}_{n+1}}(i) T \frac{1}{T} \tilde{\mathbb{E}}_i^{\mathbf{u}_{n+1}} \int_0^T \frac{g(X_t)}{h^{\mathbf{u}_{n+1}}(X_t)} dt \end{aligned} \quad (5.4)$$

if $\rho^{\mathbf{u}_{n+1}} = \rho^{\mathbf{u}_n}$. In (5.4) the expectation operator $\tilde{\mathbb{E}}_i^{\mathbf{u}_{n+1}}$ is given by

$$\tilde{\mathbb{E}}_i^{\mathbf{u}_{n+1}} f(X_t) = \mathbb{E}_i^{\mathbf{u}_{n+1}} \left[\exp \left(\theta \int_0^t (c(X_s, \mathbf{u}_{n+1}(X_s)) - \rho^{\mathbf{u}_{n+1}}) ds \right) \frac{h^{\mathbf{u}_{n+1}}(X_t)}{h^{\mathbf{u}_{n+1}}(i)} f(X_t) dt \right] \quad (5.5)$$

for any real valued bounded function f . It is easy to see that (5.5) uniquely determines a transition probability kernel $\tilde{\mathbb{P}}_i^{\mathbf{u}_{n+1}}$ and under $\tilde{\mathbb{P}}_i^{\mathbf{u}_{n+1}}$ the corresponding Markov chain is still irreducible. Since the state space is finite, the Markov chain under $\tilde{\mathbb{P}}_i^{\mathbf{u}_{n+1}}$ is positive recurrent. Thus it has a unique invariant measure, say, $\tilde{\pi}$. Now observe that the righthand side of (5.4) is negative for T sufficiently large because

$$\frac{1}{T} \tilde{\mathbb{E}}_i^{\mathbf{u}_{n+1}} \int_0^T \frac{g(X_t)}{h^{\mathbf{u}_{n+1}}(X_t)} dt \rightarrow \sum \tilde{\pi}(i) \frac{g(i)}{h^{\mathbf{u}_{n+1}}(i)} > 0.$$

But the lefthand side is always non-negative. Thus we get a contradiction and hence

$$\rho^{\mathbf{u}_{n+1}} < \rho^{\mathbf{u}_n}.$$

From the above claims it follows that the algorithm comes up with the optimal control within a finite number of steps because the number of controls is finite. \square

Some comments are now in order.

Remark 5.2. Suppose the state and action spaces are finite. Let \mathbf{u}^* be an optimal control. Let ρ^* be the optimal average cost and

$$h^{\mathbf{u}^*}(i) = \mathbb{E}_i^{\mathbf{u}^*} \left[\exp \left(\theta \int_0^{\tau_0} (c(X_t, \mathbf{u}^*(X_t)) - \rho^*) dt \right) \right].$$

Then it follows from the arguments used in the proof of Theorem 5.1 that (ρ^*, h^{u^*}) satisfies the equation

$$\theta \rho^* h^{u^*}(i) = \min_u [\theta c(i, u) h^{u^*}(i) + \sum_j \lambda_{ij}(u) h^{u^*}(j)]. \quad (5.6)$$

Equation (5.6) is the HJB equation for the average cost criterion. If (λ, h) is a solution of (5.6) where h is a positive function then using Dynkin's formula it can be shown that λ is the optimal cost and the minimiser in (5.6) is an optimal control.

Remark 5.3. If the state space is countably infinite and equation (5.6) has a solution (λ, h) such that h is a bounded, positive function which is uniformly bounded away from 0, then again it can be shown that λ is the optimal cost and the minimiser in (5.6) is an optimal control. However, we have not been able to show that (5.6) has such a solution. If one assumes that (5.6) has such a solution then one can develop value and policy iteration algorithm along the lines of [6]. In [6] the authors deal with discrete time Markov chains. There they have developed value and policy iteration algorithm under the assumption that analogous dynamic programming equation has a solution.

6. Conclusion

In this paper we have studied risk-sensitive optimal control problem for continuous time Markov chains. We have analysed the finite horizon case under fairly general conditions. For the infinite horizon discounted cost case we have assumed that the state space is finite. So it will be interesting to investigate the problem for the case of countably infinite state space. The average cost case has been studied under an additional Lyapunov type stability condition. We have established the existence of an optimal control. We have also developed policy iteration algorithm for the case of finite state and action spaces. For countable state space an algorithmic approach to determine an optimal control needs further investigation.

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